Fort Street High School

2021 ASSESSMENT TASK 4 – ONLINE EXAMINATION

Mathematics Extension 2

Total Marks: 50 Working time: 95 minutes

General Instructions

- This is an open book task
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet will be provided
- Show relevant mathematical reasoning and/or calculations

Specific Instructions:

- You will not be able to leave your desk for the duration of the task.
- Mobile phones must be turned off and out of sight.
- Your microphone and cameras must be on, but you can turn the volume on your devices down so that any noise from other students does not disturb you.
- You are not permitted to have headphones/ear buds and if you have long hair, please tie it back.
- You can ask for assistance through the direct chat function of Zoom/Teams or ask as your microphone is on.
- You are to manage your time and make sure you have a timer at hand to keep within the assessment time limit.
- At the end of the assessment, you will have 15 minutes to scan and submit your task on Google Classroom. During this time, if you have any difficulty with submitting your task, please communicate this with your teacher immediately.
- Please note that submission of the task is your responsibility.

Question 1 (16 marks) Use a new writing booklet.

(a) Let $z = \dfrac{(1+3i)}{(1-2i)}$

(i) Evaluate $(1+3i)\left(\overline{1-2i}\right)$ **2**

(ii) Hence or otherwise express z in the form of $a+ib$, where a and b are real numbers. **1**

(b) The position vectors of the points P , Q and R are given by $\overrightarrow{OP} = 2\underset{\sim}{i} + \underset{\sim}{j}$, $\overrightarrow{OQ} = -\underset{\sim}{i} + 2\underset{\sim}{k}$

and $\overrightarrow{OR} = \underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$ respectively .

(i) Find $\overrightarrow{QP}$ and $\overrightarrow{QR}$ **2**

(ii) Find projection of $\overrightarrow{QP}$ on $\overrightarrow{QR}$ **2**

(c)

(i) Find numbers A , B and C such that

$$\frac{2x+3}{(x-2)(x^2+3)} = \frac{A}{(x-2)} + \frac{Bx+C}{(x^2+3)}$$ **2**

(Show all working out)

(ii) Hence evaluate $\displaystyle\int \frac{2x+3}{(x-2)(x^2+3)}dx$ **2**

(d) Let $\underset{\sim}{u} = 2x^2\underset{\sim}{i} + 4x\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{v} = 7\underset{\sim}{i} - 2\underset{\sim}{j} + x\underset{\sim}{k}$ be given vectors. **2**

Find the value of x for which the angle between the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ is obtuse.

(e) A particle is oscillating in simple harmonic motion with period $\dfrac{\pi}{4}$. **3**

Initially, the particle has its maximum displacement of $2\sqrt{2}$ from the central point of motion.

The central point of motion of the particle is at $x = \sqrt{2}$.

Find the equation for the displacement, x of the particle in terms of t.

Question 2 (18 marks) Use a separate writing booklet.

(a) (i) Find the equation of the line l though the point $\begin{bmatrix} 2 \\ 9 \\ 13 \end{bmatrix}$ and parallel to the line passing through the points $A\ (3,-1,2)$ and $B\ (4,1,5)$. **1**

(iii) Hence, find the value of a so that the line l intersects another line m.
The equation for the line m is given below

$$\underset{\sim}{r} = a\underset{\sim}{i} + 7\underset{\sim}{j} - 2\underset{\sim}{k} + \mu(-\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}) \ .$$ **3**

(b) The acceleration of a particle P moving in a straight line is given by $a = -x^{-2}$, $x \neq 0$, x is its distance from the origin O .

The particle is initially at rest at a point P which is at a distance of 6 *metres* to the right of O.
The particle then moves in a straight line towards O.

(i) Show that $\dfrac{dx}{dt} = -\sqrt{2}\sqrt{\dfrac{6-x}{6x}}$ **3**

(ii) Using the substitution $x = 6\cos^2\theta$, show that the time taken to reach a distance 3 metres to the right of O can be given by

$$t = 12\sqrt{3}\int_0^{\frac{\pi}{4}} \cos^2\theta\, d\theta$$ **3**

(iii) Hence find t, the time taken to reach a distance 3 metres to the right of O in exact form. **2**

(c) Suppose that $I_n = \int_0^1 (x^2-1)^n\,dx$ for $n = 0,\ 1\ ,\ 2\ ,\ \ldots\ldots$

(i) Show that $I_n = \dfrac{-2n}{2n+1} I_{n-1}$, $n =\ 1\ ,\ 2\ ,\ \ldots\ldots$ **3**

(ii) Use the reduction formula to prove that $I_n = \dfrac{(-1)^n 2^{2n} (n!)^2}{(2n+1)!}$ **3**

Question 3. (16 marks) Use a separate writing booklet.

(a) Evaluate **3**

$$\int_4^7 \frac{2x-9}{2(x-3)\sqrt{x-3}}dx$$

(b) A particle is projected from the origin with velocity vector $\underset{\sim}{v} = 15(\hat{i} + \sqrt{3}\hat{j})$.

The particle experiences the effect of gravity g where $g = 10\ m/s^2$ and air resistance.

The equations of its motion in horizontal and vertical directions are given by

$\ddot{x} = -0.4\dot{x}$ and $\ddot{y} = -g - 0.4\dot{y}$

(You are NOT required to prove this)

(i) Show that $\dot{x} = 15e^{-0.4t}$ and $\dot{y} = 5\left((5+3\sqrt{3})e^{-0.4t} - 5\right)$ **3**

(ii) Show that the particle reaches its greatest height when $t = \frac{5}{2}\ln\left(\frac{5+3\sqrt{3}}{5}\right)$ **2**

(iii) Find correct to the nearest metre, the horizontal distance travelled when the particle reaches its greatest height. **2**

(c) Let w be a non-real cube root of unity.

(i) Show that $1 + w + w^2 = 0$ **2**

(ii) Show that $(1+w)^3 = -1$ **1**

(iii) Hence show that **3**

$${}^{3n}C_0 - \frac{1}{2}\left({}^{3n}C_1 + {}^{3n}C_2\right) + {}^{3n}C_3 - \frac{1}{2}\left({}^{3n}C_4 + {}^{3n}C_5\right) + {}^{3n}C_6 - \ldots\ldots\ldots\ldots\ldots + {}^{3n}C_{3n} = (-1)^n$$

End of examination

HSC Trial 2021 Solutions
Extension 2 Maths

Q1. (a) Let $z = \dfrac{1+3i}{1-2i}$

(i) $(1+3i)\,\overline{(1-2i)} = (1+3i)(1+2i)$ ✓

$= -5+5i$ ✓

(ii) $z = \dfrac{1+3i}{1-2i} = \dfrac{1+3i}{1-2i} \times \dfrac{1+2i}{1+2i} = \dfrac{-5+5i}{5}$

$= -1+i$ ✓

(b) $\overrightarrow{OP} = 2\underset{\sim}{i} + \underset{\sim}{j}$, $\overrightarrow{OQ} = -\underset{\sim}{i} + 2\underset{\sim}{k}$, $\overrightarrow{OR} = \underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$

(i) $\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ} = 3\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$ ✓

$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = 2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$ ✓

(ii) $\text{Proj}_{\overrightarrow{QR}}\, \overrightarrow{QP} = \dfrac{\overrightarrow{QP} \cdot \overrightarrow{QR}}{\overrightarrow{QR} \cdot \overrightarrow{QR}}\, \overrightarrow{QR}$

$= \dfrac{6+1-2}{6}\,(2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k})$

$= \dfrac{5}{6}\,(2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k})$

$\boxed{\checkmark \text{ Progress}}$ $\boxed{\checkmark\checkmark \text{ Correct answer}}$

(C) (i) $$\frac{2x+3}{(x-2)(x^2+3)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+3}$$

$$2x+3 = A(x^2+3) + (Bx+C)(x-2)$$

Let $x=2$, $7 = 7A \Rightarrow A=1$

Comparing coefficient of x^2

$$0 = A+B \Rightarrow B = -A \Rightarrow B = -1$$

Comparing constant term

$$3 = 3A - 2C \Rightarrow 3-3 = -2C$$

$$C = 0$$

$$\boxed{A=1, \quad B=-1, \quad C=0}$$

✓ Progress ✓✓ For correct values of all three

(ii) $\int \frac{2x+3}{(x-2)(x^2+3)}\,dx = \int \left(\frac{1}{x-2} + \frac{-x}{x^2+3}\right) dx$

$= \int \frac{1}{x-2}\,dx - \frac{1}{2}\int \frac{2x}{x^2+3}\,dx$

$= \ln|x-2| - \frac{1}{2}\ln|x^2+3| + c$

✓ one integral is correct ✓✓ Correct answer

(d) $\underset{\sim}{u} \cdot \underset{\sim}{v} < 0$ ✓

$(2x^2\underset{\sim}{i} + 4x\underset{\sim}{j} + \underset{\sim}{k}) \cdot (7\underset{\sim}{i} - 2\underset{\sim}{j} + x\underset{\sim}{k}) < 0$

$14x^2 - 8x + x < 0$

$14x^2 - 7x < 0$

$2x^2 - x < 0$

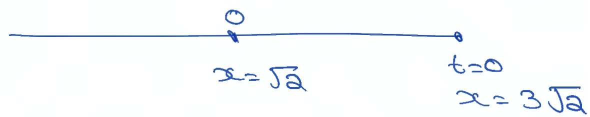
$2x(x - \frac{1}{2}) < 0$

$x(x - \frac{1}{2}) < 0$

$0 < x < \frac{1}{2}$ ✓

(e) $T = \frac{\pi}{4}$, $T = \frac{2\pi}{n} \Rightarrow n = \frac{2\pi}{T}$

$$= \frac{2\pi}{\frac{\pi}{4}} = 8$$



$a = 2\sqrt{2}$, $x_0 = \sqrt{2}$

$$x = x_0 + a \cos nt$$

$$= \sqrt{2} + 2\sqrt{2} \cos 8t$$

Q2.
(a) (i) The equation of l

$$\underset{\sim}{r} = \begin{bmatrix} 2 \\ 9 \\ 13 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \checkmark$$

or $$\underset{\sim}{r} = \begin{bmatrix} 2 \\ 9 \\ 13 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$$

(ii) The equation of m

$$\underset{\sim}{r} = \begin{bmatrix} a \\ 7 \\ -2 \end{bmatrix} + \mu \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}$$

Since l intersects m

$$\therefore \begin{bmatrix} 2+\lambda \\ 9+2\lambda \\ 13+3\lambda \end{bmatrix} = \begin{bmatrix} a-\mu \\ 7+2\mu \\ -2-3\mu \end{bmatrix} \checkmark$$

$2+\lambda = a-\mu$ ①

$9+2\lambda = 7+2\mu \Rightarrow 2\mu - 2\lambda = 2$

$\mu - \lambda = 1$ ②

$13+3\lambda = -2-3\mu \Rightarrow 3\mu + 3\lambda = -15$

$\mu + \lambda = -5$ ③

From ② and ③ $2\mu = -4 \Rightarrow \mu = -2$, $\lambda = -3$ ✓

Now using ①

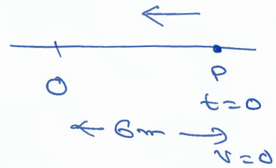
$$2+\lambda = a-\mu$$

$$2-3 = a+2$$

$$-1 = a+2$$

$$a = -3 \checkmark$$

(b) $a = -x^{-2}$



$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{1}{x^2}$$ ✓

$$\left[\frac{1}{2}v^2\right]_0^v = -\int_6^x \frac{1}{x^2}\,dx$$

$$\frac{1}{2}\left[v^2 - 0^2\right] = -\left[\frac{x^{-1}}{-1}\right]_6^x$$

$$\frac{1}{2}v^2 = \left[\frac{1}{x} - \frac{1}{6}\right] = \frac{6-x}{6x}$$

$$v^2 = 2\left(\frac{6-x}{6x}\right)$$ ✓

$$v = \pm\sqrt{2}\sqrt{\frac{6-x}{6x}}$$

Since v is -ive — as displacement is decreasing with increase in time.

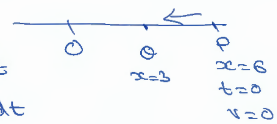
$$\therefore\ v = -\sqrt{2}\sqrt{\frac{6-x}{6x}}$$ ✓

$$\therefore\ \frac{dx}{dt} = -\sqrt{2}\sqrt{\frac{6-x}{6x}}$$

[For the above 1 mark, all three steps should be shown]

(ii)

$$\frac{dx}{dt} = -\sqrt{2}\sqrt{\frac{6-x}{6x}}$$



$$\int_6^3 \sqrt{\frac{6x}{6-x}}\,dx = -\sqrt{2}\int_0^t dt$$

or $$-\sqrt{2}\int_0^t dt = \int_6^3 \sqrt{\frac{6x}{6-x}}\,dx$$

$$\Rightarrow \quad -\sqrt{2}\left[t\right]_0^t = \int_6^3 \sqrt{\frac{6x}{6-x}}\,dx$$

$$\Rightarrow \quad t = -\frac{1}{\sqrt{2}}\int_6^3 \sqrt{\frac{6x}{6-x}}\,dx$$ ✓

let $x = 6\cos^2\theta \Rightarrow dx = 6 \times 2 \times \cos\theta \times -\sin\theta\, d\theta$
$= -12\cos\theta\sin\theta\, d\theta$

$x = 6 \Rightarrow 6 = 6\cos^2\theta \Rightarrow \cos^2\theta = 1$
$\Rightarrow \theta = 0$

$x = 3 \Rightarrow 3 = 6\cos^2\theta \Rightarrow \cos^2\theta = \frac{1}{2}$
$\cos\theta = \frac{1}{\sqrt{2}}$
$\theta = \frac{\pi}{4}$ ✓

$$\therefore \; t = \frac{-1}{\sqrt{2}}\int_0^{\frac{\pi}{4}} \sqrt{\frac{6 \times 6\cos^2\theta}{6 - 6\cos^2\theta}} \times -12\cos\theta\sin\theta\, d\theta$$

$$= \frac{-1}{\sqrt{2}}\int_0^{\frac{\pi}{4}} \sqrt{\frac{6\cos^2\theta}{\sin^2\theta}} \times -12\cos\theta\sin\theta\, d\theta$$

$$= \frac{12 \times \sqrt{6}}{\sqrt{2}}\int_0^{\frac{\pi}{4}} \frac{\cos\theta}{\cancel{\sin\theta}} \times \cos\theta\,\cancel{\sin\theta}\, d\theta$$

$$= 12\sqrt{3}\int_0^{\frac{\pi}{4}} \cos^2\theta\, d\theta$$ ✓

(iii) $t = 12\sqrt{3} \int_0^{\frac{\pi}{4}} \cos^2\theta \, d\theta$

$= 12\sqrt{3} \int_0^{\frac{\pi}{4}} \frac{1+\cos 2\theta}{2} \, d\theta$ ✓

$= 6\sqrt{3} \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\frac{\pi}{4}}$

$= 6\sqrt{3} \left[\left(\frac{\pi}{4} + \frac{1}{2} \right) - (0+0) \right]$

$= 6\sqrt{3} \left[\frac{\pi+2}{4} \right]$

$= \frac{3\sqrt{3}}{2} (\pi + 2)$ units ✓

(c) $$I_n = \int_0^1 (x^2-1)^n \, dx$$

$$= \left[(x^2-1)^n \times x\right]_0^1 - \int_0^1 n\,(x^2-1)^{n-1} \times 2x \times x \, dx$$

$$= (0-0) - 2n\int_0^1 x^2 (x^2-1)^{n-1} \, dx$$

$$= -2n \int_0^1 x^2 (x^2-1)^{n-1} \, dx$$

$$= -2n \int_0^1 (x^2-1+1)(x^2-1)^{n-1} \, dx$$

$$= -2n \int_0^1 (x^2-1)^n \, dx - 2n \int_0^1 (x^2-1)^{n-1} \, dx$$

$$= -2n\, I_n - 2n\, I_{n-1}$$

$$I_n + 2n\, I_n = -2n\, I_{n-1}$$

$$(2n+1) I_n = -2n\, I_{n-1}$$

$$I_n = \frac{-2n}{2n+1} I_{n-1}$$

(ii) $I_n = -\frac{2n}{2n+1} I_{n-1}$

$$= \frac{-2n}{2n+1} \times \frac{-2(n-1)}{2(n-1)+1} \times \frac{-2(n-2)}{2(n-2)+1} \times \cdots \times \frac{-2\times3}{2\times3+1} \times \frac{-2\times2}{2\times2+1} \times \frac{-2\times1}{2\times1+1} I_0$$

$$= \frac{(-1)^n \times 2^n \, n!}{(2n+1)(2n-1)(2n-3)\cdots 7\times5\times3} \times 1$$

$\left[\because I_0 = \int_0^1 1\,dx = [x]_0^1 = 1 \right]$

$$= \frac{(-1)^n \times 2^n \times n! \times 2n \times (2n-2)(2n-4) \times \cdots 6\times4\times2}{(2n+1)\times 2n \times (2n-1)(2n-2) \cdots 7\times6\times5\times4\times3\times2\times1}$$

$$= \frac{(-1)^n \times 2^n \times n! \times 2^n \, n(n-1)(n-2)\cdots 1}{(2n+1)!}$$

$$= \frac{(-1)^n \times 2^n \times n! \times 2^n \times n!}{(2n+1)!}$$

$$= \frac{(-1)^n \times 2^{2n} \times (n!)^2}{(2n+1)!}$$

Q3. (e) $\displaystyle\int_4^7 \frac{2x-9}{2(x-3)\sqrt{x-3}}\,dx$

$$= \int_4^7 \frac{2x-6-3}{2(x-3)\sqrt{x-3}}\,dx$$

$$= \int_4^7 \frac{2(x-3)-3}{2(x-3)\sqrt{x-3}}\,dx \quad ✓$$

$$= \int_4^7 (x-3)^{-\frac{1}{2}}\,dx - \frac{3}{2}\int_4^7 (x-3)^{-\frac{3}{2}}\,dx$$

$$= \left[\frac{(x-3)^{\frac{1}{2}}}{\frac{1}{2}}\right]_4^7 - \frac{3}{2}\left[\frac{(x-3)^{-\frac{1}{2}}}{-\frac{1}{2}}\right]_4^7 \quad ✓$$

$$= 2\left[\sqrt{x-3}\right]_4^7 + 3\left[\frac{1}{\sqrt{x-3}}\right]_4^7$$

$$= 2\left[2-1\right] + 3\left[\frac{1}{2}-1\right]$$

$$= 2 - 3\times\frac{1}{2} = 2-\frac{3}{2} = \frac{1}{2} \quad ✓$$

(b) $\underset{\sim}{v} = 15(\underset{\sim}{i} + \sqrt{3}\,\underset{\sim}{j})$

(i) $\ddot{x} = -0.4\dot{x}$

$$\frac{d\dot{x}}{dt} = -0.4\dot{x}$$

$$\int_{15}^{\dot{x}} \frac{d\dot{x}}{\dot{x}} = -\int_{0}^{t} 0.4\, dt$$

$$\left[\ln \dot{x}\right]_{15}^{\dot{x}} = -\left[0.4t\right]_{0}^{t}$$

$$\ln\dot{x} - \ln 15 = -0.4t$$

$$\ln \frac{\dot{x}}{15} = -0.4t$$

$$\frac{\dot{x}}{15} = e^{-0.4t}$$

$$\dot{x} = 15e^{-0.4t}$$ ✓

$$\ddot{y} = -g - 0.4\dot{y}$$

$$\frac{d\dot{y}}{dt} = -(g + 0.4\dot{y})$$

$$\frac{d\dot{y}}{dt} = -(10 + 0.4\dot{y})$$

$$\int_{15\sqrt{3}}^{\dot{y}} \frac{d\dot{y}}{10+0.4\dot{y}} = -\int_{0}^{t} dt$$

$$\frac{1}{0.4}\Big[\ln|10+0.4\dot{y}|\Big]_{15\sqrt{3}}^{\dot{y}} = -t$$

$$\ln|10+0.4\dot{y}| - \ln|10+0.4\times 15\sqrt{3}| = 0.4t$$

$$\ln\left|\frac{10+0.4\dot{y}}{10+6\sqrt{3}}\right| = -0.4t$$

$$\frac{10+0.4\dot{y}}{10+6\sqrt{3}} = e^{-0.4t}$$

$$10+0.4\dot{y} = (10+6\sqrt{3})\,e^{-0.4t}$$

$$0.4\dot{y} = (10+6\sqrt{3})\,e^{-0.4t} - 10$$

$$\dot{y} = \frac{1}{0.4}\left[2(5+3\sqrt{3})\,e^{-0.4t} - 10\right]$$

$$= \frac{5}{\cancel{2}} \times \cancel{2}\left\{(5+3\sqrt{3})\,e^{-0.4t} - 5\right]$$

$$= 5\left[(5+3\sqrt{3})\,e^{-0.4t} - 5\right]$$

(ii) $\dot{y} = 0$ ✓

$$5\left[(5+3\sqrt{3})e^{-0.4t} - 5\right] = 0$$

$$(5+3\sqrt{3})e^{-0.4t} = 5$$

$$e^{-0.4t} = \frac{5}{5+3\sqrt{3}}$$

$$-0.4t = \ln \frac{5}{5+3\sqrt{3}}$$

$$t = \frac{-1}{0.4} \ln \frac{5}{5+3\sqrt{3}}$$

$$= \frac{5}{2} \ln\left(\frac{5+3\sqrt{3}}{5}\right)$$ ✓

(iii) $\dot{x} = 15e^{-0.4t}$

$$\frac{dx}{dt} = 15e^{-0.4t}$$

$$\int_0^x dx = 15\int_0^t e^{-0.4t}\,dt$$

$$x = 15\left[\frac{e^{-0.4t}}{-0.4}\right]_0^t$$

$$x = -15 \times \frac{5}{2}\left[e^{-0.4t} - 1\right]$$

$$x = \frac{-75}{2}\left[e^{-0.4t} - 1\right]$$ ✓

When $t = \frac{5}{2} \ln \frac{5+3\sqrt{3}}{5}$

$$e^{-0.4t} = e^{-\frac{\cancel{2}}{\cancel{5}} \times \frac{\cancel{5}}{\cancel{2}} \ln\left(\frac{5+3\sqrt{3}}{5}\right)}$$

$$= \frac{5}{5+3\sqrt{3}}$$

$$\therefore x = \frac{-75}{2}\left[\frac{5}{5+3\sqrt{3}} - 1\right]$$

$$\approx 19 \text{ m}$$ ✓

(iii) $(1+\omega)^{3n} = \left((1+\omega)^3\right)^n = (-1)^n$ ①

[Using Part (ii)]

Using Binomial Theorem ✓

$(1+\omega)^{3n}$

$= {}^{3n}C_0 + {}^{3n}C_1\omega + {}^{3n}C_2\omega^2 + {}^{3n}C_3\omega^3 + {}^{3n}C_4\omega^4$
$+ {}^{3n}C_5\omega^5 + {}^{3n}C_6\omega^6 + \cdots\cdots + {}^{3n}C_{3n}\omega^{3n}$

$= {}^{3n}C_0 + {}^{3n}C_1\omega + {}^{3n}C_2\omega^2 + {}^{3n}C_3 + {}^{3n}C_4\omega$
$+ {}^{3n}C_5\omega^2 + {}^{3n}C_6 + \cdots\cdots + {}^{3n}C_{3n}\omega^{3n}$ ②

Now $1 + \omega + \omega^2 = 0$

or $\omega^2 + \omega + 1 = 0$

$\omega = \dfrac{-1 \pm \sqrt{1-4}}{2} = \dfrac{-1 \pm \sqrt{3}i}{2}$

$\therefore \omega = \dfrac{-1+\sqrt{3}i}{2}$ or $\omega = \dfrac{-1-\sqrt{3}i}{2}$

If $\omega = \dfrac{-1+\sqrt{3}i}{2}$

$\omega^2 = \left(\dfrac{-1+\sqrt{3}i}{2}\right)^2 = \dfrac{-1-3-2\sqrt{3}i}{4}$

$= \dfrac{-4-2\sqrt{3}i}{4}$

$= \dfrac{-1-\sqrt{3}i}{2}$

Now from ② and ①

$$(1+\omega)^{3n} = {}^{3n}C_0 + {}^{3n}C_1\left(\frac{-1+\sqrt{3}i}{2}\right) + {}^{3n}C_2\left(\frac{-1-\sqrt{3}i}{2}\right)$$
$$+ {}^{3n}C_3 + {}^{3n}C_4\left(\frac{-1+\sqrt{3}i}{2}\right) + {}^{3n}C_5\left(\frac{-1-\sqrt{3}i}{2}\right)$$
$$+ {}^{3n}C_6 + \cdots + {}^{3n}C_{3n}$$

$$= {}^{3n}C_0 + {}^{3n}C_1\left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right) + {}^{3n}C_2\left(\frac{-1}{2} - \frac{\sqrt{3}}{2}i\right)$$
$$+ {}^{3n}C_3 + {}^{3n}C_4\left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right) + {}^{3n}C_5\left(\frac{-1}{2} - \frac{\sqrt{3}}{2}i\right)$$
$$+ {}^{3n}C_6 + \cdots + {}^{3n}C_{3n}$$

$$\therefore\ {}^{3n}C_0 + {}^{3n}C_1\left(\frac{-1}{2}\right) + {}^{3n}C_1\left(\frac{\sqrt{3}}{2}i\right) + {}^{3n}C_2\left(\frac{-1}{2}\right)$$
$$+ {}^{3n}C_2\left(-\frac{\sqrt{3}}{2}i\right) + {}^{3n}C_3 + {}^{3n}C_4\left(\frac{-1}{2}\right)$$
$$+ {}^{3n}C_4\left(\frac{\sqrt{3}}{2}i\right) + {}^{3n}C_5\left(\frac{-1}{2}\right)$$
$$+ {}^{3n}C_5\left(-\frac{\sqrt{3}}{2}i\right) + {}^{3n}C_6 + \cdots + {}^{3n}C_{3n}$$
$$= (-1)^n \quad ✓$$

Comparing Real parts on both sides

$${}^{3n}C_0 + {}^{3n}C_1\left(\frac{-1}{2}\right) + {}^{3n}C_2\left(\frac{-1}{2}\right) + {}^{3n}C_3$$
$$+ {}^{3n}C_4\left(\frac{-1}{2}\right) + {}^{3n}C_5\left(\frac{-1}{2}\right) + {}^{3n}C_6 + \cdots$$
$$\cdots + {}^{3n}C_{3n} = (-1)^n$$

$$\therefore\ {}^{3n}C_0 - \frac{1}{2}\left({}^{3n}C_1 + {}^{3n}C_2\right) + {}^{3n}C_3$$
$$- \frac{1}{2}\left({}^{3n}C_4 + {}^{3n}C_5\right) + \cdots + {}^{3n}C_{3n}$$
$$= (-1)^n \quad ✓$$

End of Solutions

🙂